

Breaking Down An Equation

Notes by Ben Ong

I. INTRODUCTION

CHALLENGE: Solve the following PDE's:

$$u_t + V(x)u_x + V_x(x)u = 0, \tag{1a}$$

$$u_t + V(x)u_x + V_x(x)u = -\frac{1}{2}(V^2(x))_x, \tag{1b}$$

with corresponding initial conditions

$$u(x, 0) = V(x) = -\epsilon \sin x, \tag{2a}$$

$$u(x, 0) = 0. \tag{2b}$$

They look (and are) intimidating to solve. However, if we simplify and work on parts of the equation, we may gain useful insight to solving the full equations.

II. SIMPLEST CASE

We start by considering only the first two terms of Equation 1a

$$u_t + V(x)u_x = 0, \tag{3a}$$

$$u(x, 0) = V(x) = -\epsilon \sin(x). \tag{3b}$$

If we assume a solution of the form $u(x(s), t(s))$, then

$$\frac{du}{ds} = \frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial t} \frac{dt}{ds}. \tag{4}$$

Comparing terms with Equation 3a, we have

$$\frac{du}{ds} = 0, \tag{5}$$

i.e., along characteristics, the solution is constant, provided

$$\frac{dx}{ds} = V(x) = -\epsilon \sin x, \quad \frac{dt}{ds} = 1. \tag{6}$$

Integrating Equation 6, we get

$$s = t + c_1, \tag{7a}$$

$$\begin{aligned} \ln |\csc x - \cot x| &= -\epsilon s + c_2, \\ &= -\epsilon t + c. \end{aligned} \tag{7b}$$

We can simplify Equation 7b. Consider

$$\begin{aligned} \csc x - \cot x &= \frac{1 - \cos x}{\sin x} = \frac{1 - \cos x}{\sin x} \left(\frac{1 + \cos x}{1 + \cos x} \right), \\ &= \frac{1 - \cos^2 x}{\sin x(1 + \cos x)} = \frac{\sin x}{1 + \cos x}, \\ &= \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}}, \\ &= \tan \frac{x}{2}. \end{aligned}$$

Thus, Equation 7b simplifies to

$$c = \ln \left| \tan \frac{x}{2} \right| + \epsilon t, \tag{8}$$

and the general solution for Equation 3a is

$$u(x, t) = f \left(\ln \left| \tan \frac{x}{2} \right| + \epsilon t \right). \tag{9}$$

We still need to satisfy the given initial condition,

$$u(x, 0) = f \left(\ln \left| \tan \frac{x}{2} \right| \right) = -\epsilon \sin x. \tag{10}$$

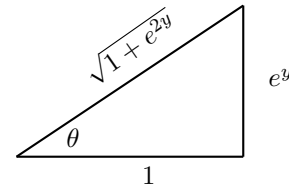
We choose

$$\begin{aligned} y = \ln \left(\tan \frac{x}{2} \right) &\implies e^y = \tan \frac{x}{2}, \\ &\implies x = 2 \arctan(e^y), \end{aligned}$$

where we have dropped $|\cdot|$ for ease of notation. Therefore

$$f(y) = -\epsilon \sin(2 \arctan(e^y)). \tag{11}$$

To further simplify Equation 11, let $\theta = \arctan(e^y)$. Since $\tan \theta = e^y$,



$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta, \\ &= 2 \left(\frac{e^y}{\sqrt{1 + e^{2y}}} \right) \left(\frac{1}{\sqrt{1 + e^{2y}}} \right), \\ &= 2 \frac{e^y}{1 + e^{2y}}, \\ &= \frac{2}{e^{-y} + e^y}, \\ &= \operatorname{sech} y. \end{aligned}$$

Substituting back into Equation 11

$$f(y) = -\epsilon \operatorname{sech} y. \tag{12}$$

So our solution is:

$$\begin{aligned} u(x, t) &= f\left(\ln\left(\tan\frac{x}{2}\right) + \epsilon t\right), \\ &= -\epsilon \operatorname{sech}\left(\ln\left(\tan\frac{x}{2}\right) + \epsilon t\right), \\ &= \frac{-2\epsilon}{e^{\epsilon t} \tan\frac{x}{2} + e^{-\epsilon t} \cot\frac{x}{2}}, \\ &= \frac{-2\epsilon \sin\frac{x}{2} \cos\frac{x}{2}}{e^{\epsilon t} \sin^2\frac{x}{2} + e^{-\epsilon t} \cos^2\frac{x}{2}}, \\ &= \frac{-\epsilon \sin x}{e^{\epsilon t} \left(\frac{1-\cos x}{2}\right) + e^{-\epsilon t} \left(\frac{1+\cos x}{2}\right)}, \\ &= \frac{-\epsilon \sin x}{\frac{1}{2}(e^{-\epsilon t} + e^{\epsilon t}) - \frac{1}{2} \cos x (e^{\epsilon t} - e^{-\epsilon t})}, \\ &= \frac{-\epsilon \sin x}{\cosh \epsilon t - \cos x \sinh \epsilon t}. \end{aligned}$$

Isn't it fantastic that after all this algebra, we are left with such a simple looking solution?

III. SOLVING EQUATION 1a

$$u_t + V(x) u_x + V_x(x) u = 0.$$

Equation 1a is really a simple variant of the previous problem. Combining the derivatives and rewriting, we have

$$u_t + (V(x)u)_x = 0.$$

Multiplying both sides by $V(x)$,

$$V(x)u_t + V(x)(V(x)u)_x = 0.$$

Substituting $w(x, t) = V(x)u(x, t)$, we get

$$w_t + V(x)w_x = 0, \tag{13a}$$

$$w(x, 0) = V(x)u(x, 0) = V^2(x) = \epsilon^2 \sin^2 x. \tag{13b}$$

We follow the steps and arguments in the previous section until Equation 10.

$$w(x, 0) = \epsilon^2 \sin^2 x = f(y).$$

By the same arguments,

$$f(y) = \epsilon^2 \operatorname{sech}^2 y.$$

The solution is:

$$w(x, t) = \frac{\epsilon^2 \sin^2 x}{(\cosh \epsilon t - \cos x \sinh \epsilon t)^2},$$

or

$$u(x, t) = \frac{-\epsilon \sin x}{(\cosh \epsilon t - \cos x \sinh \epsilon t)^2}.$$

There may be other simplifications, but I was too lazy to go through the calculations and check.

IV. SOLVING EQUATION 1b

$$u_t + V(x) u_x + V_x(x) u = -\frac{1}{2} (V^2(x))_x,$$

Following the same trick as Section III, we multiply both sides by $V(x)$ and substitute $w(x, t) = V(x) u(x, t)$. This results in the following set of equations:

$$w_t + V w_x = \frac{1}{2} (V^2)_x V, \tag{14a}$$

$$w(x, 0) = 0. \tag{14b}$$

If we assume a solution of the form $w(x(s), t(s))$, then

$$\frac{dw}{ds} = \frac{1}{2} (V^2)_x V, \tag{15}$$

and the characteristic equations remain the same as before.

$$\frac{dx}{ds} = V(x), \quad \frac{dt}{ds} = 1.$$

Giving rise to the characteristic equations

$$c_1 = t - s,$$

$$c_2 = \ln \left| \tan \frac{x}{2} \right| + \epsilon s.$$

We pick c_1 such that when $s = 0$, then $t = 0$. (i.e. pick $c_1 = 0$). We pick c_2 such that when $s = 0$, then $x = x_0$, i.e.

$$c_2 = \ln \left| \tan \frac{x_0}{2} \right|. \tag{16}$$

Let me emphasize that given (x, t) , we can solve for (x_0, s) so that we know which characteristic we are on. Now, along the characteristic,

$$\frac{dw}{ds} = \frac{1}{2} (V^2)_x V.$$

Changing variables,

$$\frac{dw}{dx} \frac{dx}{ds} = \frac{1}{2} (V^2)_x V,$$

$$\frac{dw}{dx} = \frac{1}{2} \frac{(V^2)_x V}{V} = \frac{1}{2} (V^2)_x.$$

Then,

$$\int_{x_0}^x \frac{dw}{dx} dx = \int_{x_0}^x \frac{1}{2} (V^2)_x dx,$$

$$w(x, t) - w(x_0, t) = \frac{1}{2} V^2(x) - \frac{1}{2} V^2(x_0).$$

Since $x = x_0 \implies t = 0 \implies w(x_0, t) = 0$,

$$w(x, t) = \frac{1}{2} V^2(x) - \frac{1}{2} V^2(x_0),$$

$$u(x, t) = -\frac{1}{2} \epsilon \sin x + \frac{1}{2} \frac{\epsilon \sin^2 x_0}{\sin x}. \tag{17}$$

where

$$\begin{aligned} \ln \left(\tan \frac{x_0}{2} \right) &= \ln \left(\tan \frac{x}{2} \right) + \epsilon t, \\ x_0 &= 2 \arctan \left(e^{\epsilon t} \tan \frac{x}{2} \right). \end{aligned} \tag{18}$$

Using tricks from Section II,

$$\begin{aligned} \sin\left(2 \arctan\left(e^{\epsilon t} \tan \frac{x}{2}\right)\right) &= \frac{2e^{\epsilon t} \tan \frac{x}{2}}{e^{2\epsilon t} \tan^2 \frac{x}{2} + 1}, \\ &= \frac{\sin x}{e^{\epsilon t} \sin^2 \frac{x}{2} + e^{-\epsilon t} \cos^2 \frac{x}{2}}, \\ &= \frac{\sin x}{\cosh \epsilon t - \cos x \sinh \epsilon t}. \end{aligned}$$

Thus, our final solution becomes

$$\begin{aligned} u(x, t) &= -\frac{1}{2}\epsilon \sin x + \frac{\epsilon}{2} \frac{\sin x}{(\cosh \epsilon t - \cos x \sinh \epsilon t)^2} \\ &= -\frac{\epsilon \sin x}{2} \left(1 - \frac{1}{(\cosh \epsilon t - \cos x \sinh \epsilon t)^2}\right) \quad (19) \end{aligned}$$