$$u(x,t) = \sin x \left\{ \frac{\operatorname{sech}^2 \epsilon t}{(1-\tanh \epsilon t \cos x)^2} \right\}$$

Give its fourier series in x

a) why would it be a sine series?

b) what is the coeff integral?

c) evaluate this integral. (Hint, complex contour integration will be useful... why?)

1.) Sine Series.

note that $u(x,t)$ is <u>odd</u> in x

$$u(x,t) = \underbrace{\sin x}_{\text{odd}} \underbrace{\{ \quad \}}_{\text{even.}}$$

$\Rightarrow$ expect only sine series.

note: u 2π-periodic in x

2) coeff integral

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \dfrac{\operatorname{sech}^2 \epsilon t}{\pi} \displaystyle\int_{-\pi}^{\pi} \frac{\sin(nx) \sin x \, dx}{(1-\tanh \epsilon t \cos x)^2}$

$a_0 = \dfrac{\operatorname{sech}^2 \epsilon t}{2\pi} \displaystyle\int_{-\pi}^{\pi} \frac{\sin x \, dx}{(1-\tanh \epsilon t \cos x)^2}$

Evaluating the integral.

we are "interested" in evaluating

$$\int_{-\pi}^{\pi} \frac{\sin(nx)\sin x \, dx}{(1 - k\cos x)^2}$$

where $k < 1$ to find b_n's

it doesn't appear as though simple trig substitution will get us anywhere.
why would complex contour integration help / how could we simplify the resulting integral.

- → pick a different path of integration
- → residue theorem
- → negligible contributions from contours @ infinity
- → exploit symmetries
- → integral on a particular contour is known.
- → integral on a particular contour is constant.
- →

extending this integral to the complex plane

$$\int_{-\pi}^{\pi} \frac{\sin(nz)\sin z \, dz}{(1 - k\cos z)^2} \qquad \text{where } k \in \mathbb{R},\ k < 1.$$

what are the poles of this equation?

$$\cos z = \frac{1}{k}.$$

if $z = x + iy$

$\cos(z) = \cos(x + iy)$

$= \cos x \cosh y + i \sin x \sinh y = \frac{1}{k}$

k real $\Rightarrow$ $\sin x \sinh y \equiv 0$

either $\sin x = 0 \Rightarrow \boxed{x = 2n\pi}$ or $\sinh y = 0 \Rightarrow \boxed{y = 0}$

$$\cos x \cosh y = \frac{1}{k}.$$

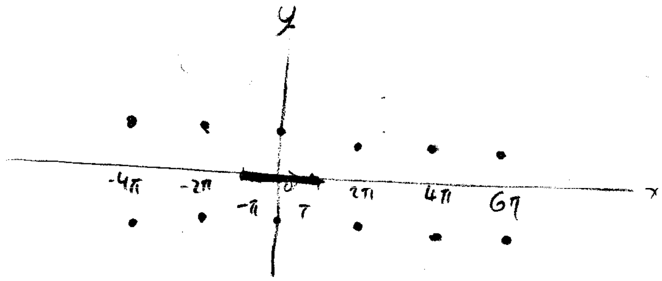
case $x = 2n\pi$, $\Rightarrow \cosh y = \frac{1}{k}$.

$$y = \pm\cosh^{-1}\left(\frac{1}{k}\right)$$

case $y = 0 \Rightarrow \cos x = \frac{1}{k}$.

but $\frac{1}{k} > 1$ $\therefore$ no such $\underline{\underline{x}}$

$\therefore$ the poles are 2π periodic

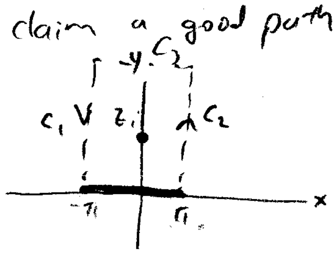


First before we discuss which paths to take, notice that we can break the integral into two parts

$$\int_{-\pi}^{\pi} \frac{\sin(nz)\sin z \, dz}{(1-k\cos z)^2} = \int_{-\pi}^{\pi} \frac{\left(e^{inz} - e^{-inz}\right)\sin z \, dz}{2i(1-k\cos z)^2}$$

$$= \overbrace{\frac{1}{2i}\int \frac{e^{inz}\sin z \, dz}{(1-k\cos z)^2}}^{①} - \overbrace{\frac{1}{2i}\int \frac{e^{-inz}\sin z \, dz}{(1-k\cos z)^2}}^{②}$$

for ① claim a good path

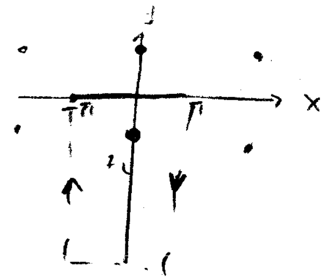


b/c. $\int_{C_1} = -\int_{C_2}$ (periodicity) & opp sign

$(y \to \infty)$ $\int_{C_3} = 0$ b/c e^{inz} term dominates,

$e^{inz} \to 0$ as $y \to \infty$

for ② claim



with similar arguments

expect ① $= 2\pi i \sum \mathrm{res}\,(z = z_1)$

② $= 2\pi i \sum \mathrm{res}\,(z = z_2$