# Level Set Framework for Image Segmentation

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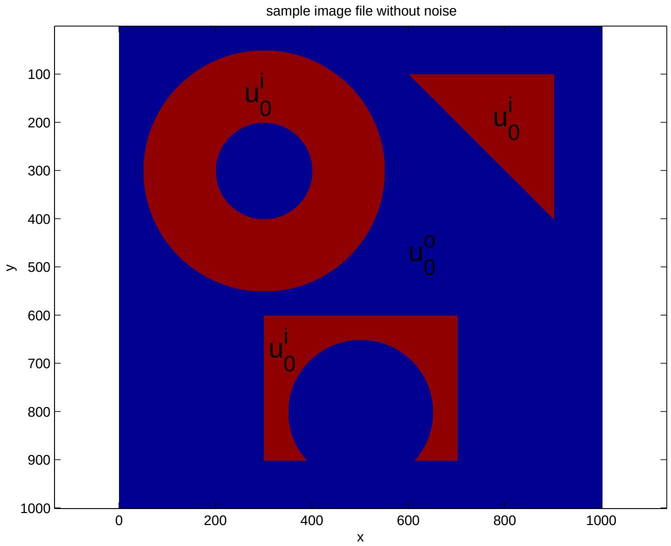
## I. Introduction

To detect objects in an image, active contour models evolve an initial curve subject to constraints specified in the image. In **[1]** T.F. Chan and L.A. Vese proposed an active contour model using an energy minimization technique. Their model works on noisy/blurred images, and does not rely on gradient values to find the boundary.

## II. Model




Assume that image $u_0$ is formed by two regions of approximately constant intensities $u_0^i$ and $u_0^o$, and the object to be detected is represented by the region with value $u_0^i$. If the boundary is given by $C_0$, then $u_0 \approx u_0^i$ inside $C_0$ and $u_0 \approx u_0^o$ outside $C_0$. The following fitting energy

$$F_1(C) + F_2(C) = \int_{\text{inside } C} |u_0 - c_1|^2 dx + \int_{\text{outside } C} |u_0 - c_2|^2 dx \quad (1)$$

(where $C$ is any variable curve, $c_1$ and $c_2$ are constants depending on $C$) is minimized when $C = C_0$. i.e.

$$\inf_C \{F_1(C) + F_2(C)\} \approx 0 \approx F_1(C_0) + F_2(C_0)$$

Adding some regularizing terms like the length of $C$ and the area inside $C$, the energy function $F(C, c_1, c_2)$ is given by

$$F(C, c_1, c_2) = \mu \text{ (length of } C)^p + \nu \text{ (area inside } C) \quad (2)$$
$$+\lambda_1 \int_{\text{inside } C} |u_0 - c_1|^2 dx + \lambda_2 \int_{\text{outside } C} |u_0 - c_2|^2 dx$$

$c_1$ and $c_2$ are constant unknowns, $\mu \geq 0$, $\nu \geq 0$, $\lambda_1, \lambda_2 > 0 p > 0$ are fixed constants.

## III. Level Set Formulation

In the level set method **[2]**, $C$ is represented by the zero level set of a Lipschitz function $\phi : \mathbf{R}^N \to \mathbf{R}$ such that

$$\begin{aligned} C &= \{x \in \mathbf{R}^N : \phi(x) = 0\} \\ \text{inside } C &= \{x \in \mathbf{R}^N : \phi(x) > 0\} \\ \text{outside } C &= \{x \in \mathbf{R}^N : \phi(x) < 0\} \end{aligned}$$

Using the standard definition for the Heaviside function $H$ and the dirac measure $\delta$

$$H(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

$$\delta(z) = \frac{d}{dz} H(z) \text{ (in the sense of distributions)}$$

We have the following results:

$$\text{length of } C = \int_\Omega |\nabla H(\phi)| dx = \int_\Omega \delta(\phi) |\nabla \phi| dx \quad (3)$$

$$\text{Area inside } C = \int_\Omega H(\phi) dx \text{ , thus} \quad (4)$$

$$\int_{\text{inside } C} |u_0 - c_1|^2 dx = \int_\Omega |u_0 - c_1|^2 H(\phi) dx \quad (5)$$

$$\int_{\text{outside } C} |u_0 - c_2|^2 dx = \int_\Omega |u_0 - c_1|^2 \{1 - H(\phi)\} dx \quad (6)$$

Therefore,

$$F(\phi, c_1, c_2) = \mu \left( \int_\Omega \delta(\phi) |\nabla \phi| dx \right)^p + \nu \int_\Omega H(\phi) dx + \quad (7)$$
$$\lambda_1 \int_\Omega |u_0 - c_1|^2 H(\phi) dx + \lambda_2 \int_\Omega |u_0 - c_1|^2 \{1 - H(\phi)\} dx$$

Minimizing the energy functional with respect to $c_1$ and $c_2$ gives

$$c_1(\phi) = \frac{\int_\Omega u_0 H(\phi) dx}{\int_\Omega H(\phi) dx} \quad (8)$$

$$c_2(\phi) = \frac{\int_\Omega u_0 \{1 - H(\phi)\} dx}{\int_\Omega \{1 - H(\phi)\} dx} \quad (9)$$

which correspond to the average value of $u_0$ inside $C$ and outside $C$ respectively. (note, the curve must have a non-empty interior and exterior).

## IV. Euler-Lagrange Equations

To compute the associated Euler-Lagrange equations for $\phi$, we need regularized versions of $H$ and $\delta$ such that $\delta_\epsilon = H'_\epsilon$ In particular, we use

$$H_\epsilon(z) = \frac{1}{2} \left( 1 + \frac{2}{\pi} \arctan \left( \frac{z}{\epsilon} \right) \right) \quad (10)$$

$$\delta_\epsilon(z) = H'_\epsilon(z) = \frac{1}{\pi} \left( \frac{\epsilon}{\epsilon^2 + z^2} \right)$$

The associated regularized functional $F_\epsilon$ of $F$ will be

$$F_\epsilon(\phi, c_1, c_2) = \mu \left( \int_\Omega \delta_\epsilon(\phi)|\nabla\phi| dx \right)^p + \nu \int_\Omega H_\epsilon(\phi) dx \quad (11)$$
$$+\lambda_1 \int_\Omega |u_0 - c_1|^2 H_\epsilon(\phi) dx + \lambda_2 \int_\Omega |u_0 - c_1|^2 \{1 - H_\epsilon(\phi)\} dx$$

Keeping $c_1$ and $c_2$ fixed we compute

$$\lim_{t\to 0} \frac{1}{t} \left[ F_\epsilon(\phi + t\psi, c_1, c_2) - F_\epsilon(\phi, c_1, c_2) \right] \quad (12)$$

(where $\psi$ is a test function) to obtain the Euler Lagrange equations for $\phi$.

$$\delta_\epsilon(\phi) \Bigg[ \mu\, p \left( \int_\Omega \delta_\epsilon(\phi)|\nabla\phi| \right)^{p-1} \nabla \cdot \left( \frac{\nabla\phi}{|\nabla\phi|} \right) \quad (13)$$
$$-\nu - \lambda_1 (u_0 - c_1)^2 + \lambda_2 (u_0 - c_2)^2 \Bigg] = 0 \text{ on } \Omega$$

$$p \left( \int_\Omega \delta_\epsilon(\phi)|\nabla\phi| dx \right)^{p-1} \frac{\delta_\epsilon(\phi)}{|\nabla\phi|} \frac{\partial\phi}{\partial n} = 0 \text{ on } \partial\Omega \quad (14)$$

## V. Femlab Implementation

1. Intialize $\phi^0$
2. Calculate $c_1(\phi^0)$, $c_2(\phi^0)$ and $L = \int_\Omega \delta_\epsilon(\phi)|\nabla\phi|\, dx$
3. Solve PDE

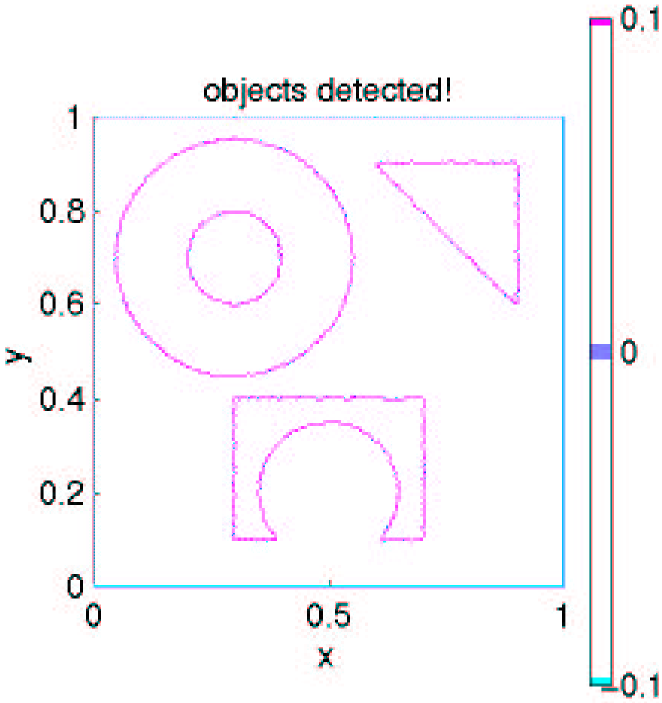
$$\frac{\partial\phi}{\partial t} = \delta_\epsilon(\phi) \Bigg[ \mu\, p(L)^{p-1} \nabla \cdot \left( \frac{\nabla\phi}{|\nabla\phi|} \right) - \nu \quad (15)$$
$$-\lambda_1 (u_0 - c_1)^2 + \lambda_2 (u_0 - c_2)^2 \Bigg] = 0 \text{ on } \Omega$$
$$\frac{\partial\phi}{\partial n} = 0 \text{ on } \partial\Omega \quad (16)$$

## VI. Comments

- Ideally, one would loop steps 2 and 3. (i.e., after every time step, recalculate $c_1$,$c_2$ and $L$) however this was not attempted because the code worked perfectly well. Presumably, one would obtain better convergence if $c_1$,$c_2$ and $L$ was recalculated on a frequent basis.
- We treated $L = \int_\Omega \delta_\epsilon(\phi)|\nabla\phi|$ explicitly because it does not pose the stiffnes contraints created by the curvature term, $\kappa = \nabla \cdot \left( \frac{\nabla\phi}{|\nabla\phi|} \right)$
- It is probably a good idea to do a reinitialization after a few steps. (For this test case, it didn't make a difference.)
- The following parameters were used for the code, $\nu = 0, \mu = 0.1, \lambda_1 = 1, \lambda_2 = 1, p = 2$

## VII. Results

Starting with an initial contours centered at $(0.5, 0.5)$ with radii 0.1, 0.3 and 0.4, the algorithm converged correctly to the the test image. It was impressive to watch the contour deal with topological changes, the sharp edges and the convex shape.




## References

[1] T.F. Chan and L.A. Vese, *Active Contours without edges*, IEEE transactions on Image Processing, 2001, 10(2): 266-277.
[2] S. Osher and J.A. Sethian *Fronts Propagating with Curvature-Dependent Speed* Journal of Computational Physics, 1988 79, pp12-49.