Staggered Grid Analysis for Shallow Water Approximation

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I. Introduction

Consider an inviscid, incompressible 2-D fluid. The shallow water equations are a coupled system of 1-D nonlinear hyperbolic equations:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}. \tag{1}$$

where $h(x,t)$ is the height of the wave, and $u(x,t)$ is the x-velocity. We assume that

$$u = U + u', \quad h = H + h',$$

where u', h' are small, U, H are constants, and linearize (1) to give

$$\frac{\partial h'}{\partial t} + U\frac{\partial h'}{\partial x} + H\frac{\partial u'}{\partial x} = 0,$$

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + g\frac{\partial h'}{\partial x} = 0. \tag{2}$$

We note that there are two physical mechanisms acting advection and gravity. Dropping the prime notation, we can isolate both effects by splitting the equation as follows:

$$\begin{aligned}\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} &= 0,\\ \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} &= 0, \quad \text{(Advection)}\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} &= 0,\\ \frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} &= 0. \quad \text{(Gravity)}\end{aligned} \tag{4}$$

We ignore the advection equation (3) and work with equation (4). We non-dimensionalize the system by making the following substitutions.

$$\tilde{u} = \frac{u}{U^*}, \quad \tilde{h} = \frac{h}{H}, \quad \tilde{x} = \frac{x}{H},$$
$$\tilde{t} = \frac{U^*}{H}t, \quad \tilde{L} = \frac{L}{H}, \quad (U^*)^2 = gH.$$

Dropping the tilde notation, we have our simplified system

$$\frac{\partial u}{\partial t} + \frac{\partial h}{\partial x} = 0,$$

$$\frac{\partial h}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{5}$$

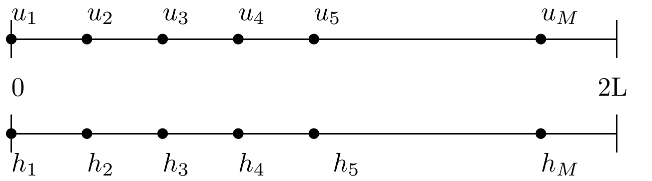

Fig. 1. Unstaggered Grid

II. Staggered Grid

We now consider the best way of distributing the variables u and h. The standard approach is to use an *unstaggered grid* as shown in Figure 1. The discretization becomes

$$\begin{aligned}\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} &= \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x},\\ \frac{h_j^{n+1} - h_j^{n-1}}{2\Delta t} &= \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}.\end{aligned} \tag{6}$$

Notice that since centered differences are used, we have *two* seperate subgrids. (i.e., the solution on the subgrids become decoupled from one another!) To alleviate this problem, we consider the staggered grid shown in Figure 2. The resulting discretization now becomes

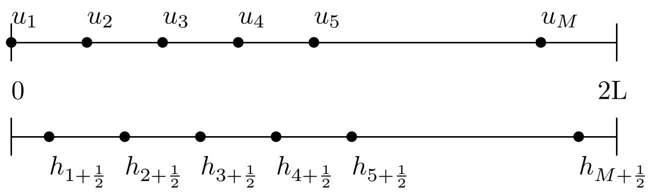

Fig. 2. Staggered Grid

$$\begin{aligned}\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} &= \frac{h_{j+1/2}^n - h_{j-1/2}^n}{\Delta x},\\ \frac{h_{j+1/2}^{n+1} - h_{j+1/2}^{n-1}}{2\Delta t} &= \frac{u_{j+1}^n - u_j^n}{\Delta x}.\end{aligned} \tag{7}$$

The advantages of such a discretization are that the centered space derivative now uses sucessive points of the same variable, and the dispersion characteristics of any scheme is improved because the effective grid length is halved.

III. Dispersion Relation

To derive the dispersion relation, we consider the discrete travelling wave

$$u_j^n = u_0 e^{i(kj\Delta x - n\omega\Delta t)},$$
$$h_j^n = h_0 e^{i(kj\Delta x - n\omega\Delta t)}. \tag{8}$$

Substituting (8) into (7), we obtain the following relations

$$u_j^n \left(\frac{e^{-i\omega t} - e^{i\omega t}}{2\Delta t} \right) + h_j^n \left(\frac{e^{\frac{ik\Delta x}{2}} - e^{-\frac{ik\Delta x}{2}}}{\Delta x} \right) = 0$$
$$h_j^n e^{\frac{ik\Delta x}{2}} \left(\frac{e^{-i\omega t} - e^{i\omega t}}{2\Delta t} \right) + u_j^n \left(\frac{e^{ik\Delta x} - 1}{\Delta x} \right) = 0$$

Cancelling common terms and simplifying yields

$$\frac{u_0}{\Delta t}(-i\sin\omega\Delta t) + \frac{h_0}{\Delta x}\left(2i\sin\frac{k\Delta x}{2}\right) = 0$$
$$\frac{h_0}{\Delta t}(-i\sin\omega\Delta t) + \frac{u_0}{\Delta x}\left(2i\sin\frac{k\Delta x}{2}\right) = 0$$

or in vector form,

$$\begin{bmatrix} -i\sin\omega\Delta t & 2i\lambda\sin\frac{k\Delta x}{2} \\ 2i\lambda\sin\frac{k\Delta x}{2} & -i\sin\omega\Delta t \end{bmatrix} \begin{bmatrix} u_0 \\ h_0 \end{bmatrix} = 0 \tag{9}$$

Taking the determinant of the matrix yields the dispersion relation,

$$-\sin^2\omega\Delta t = -4\lambda^2\sin^2\frac{k\Delta x}{2}$$
$$\sin\omega\Delta t = \pm 2\lambda\sin\frac{k\Delta x}{2} \tag{10}$$

For stability, we require $\boxed{0 < \lambda \leq 0.5}$. Assuming Δt and Δx are sufficiently small, we can taylor expand (10) and get

$$\omega\Delta t \approx \pm 2\lambda \left(\left(\frac{k\Delta x}{2}\right) - \frac{1}{3!}\left(\frac{k\Delta x}{2}\right)^3 \cdots \right)$$
$$\omega \approx \pm k\left(1 - \frac{k^2\Delta x^2}{24}\right) \tag{11}$$

Since ω is real, we conclude that there is no amplitude growth or decay. Further, since the phase error is negative, we conclude that the waves are deccelerating. We note however that the wave is decelerating much slower compared to the unstaggered grid. It can be shown that for the *unstaggered* grid,

$$\omega \approx \pm k\left(1 - \frac{k^2\Delta x^2}{6}\right) \tag{12}$$

IV. Numerical Evidence

We began by checking the stability condition. As expected, satisfying the CFL condition $\lambda \leq 0.5$ resulted in a stable solution, whereas $\lambda > 0.5$ resulted in an unstable growth. We also present plots the following plots confirming our other results.

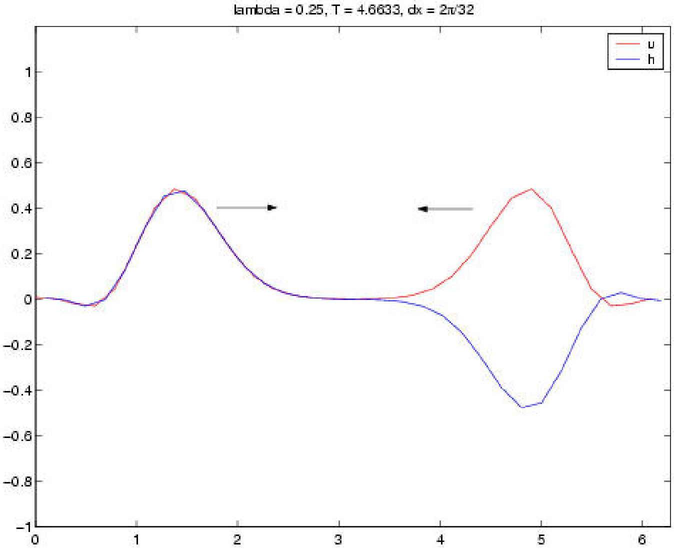

Fig. 3. Snapshot of wave evolution at T = 4.66. Notice that the higher wave numbers are lagging, confirming the decceleration. We had to reduce Δx in order to observe this phenomenon.

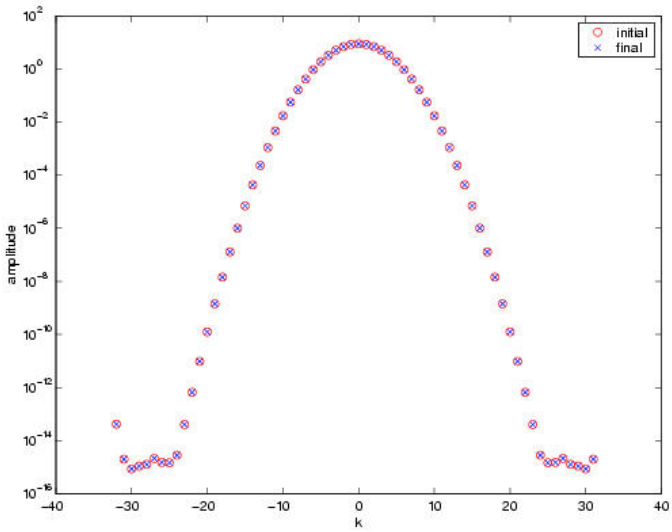

Fig. 4. A plot of fourier coefficients at time $t = 0$ and $T = 2\pi$. Convincing evidence that there is no decay or growth in amplitude! $\lambda = \frac{1}{2}$, $dx = \frac{2\pi}{64}$.

References

[1] D.R. Duran, *Numerical Methods for Wave Equations in Geophysical Fluid Dynamics*. New York: Springer, 1999.
[2] M. Hortal, "Numerical Methods", *"http://www.ecmwf.int/newsevents/training/rcourse_notes"*, Mar 2001